13

VACUUM SOLUTIONS WITH NON-ALIGNED POLARIZATION

The purpose of this chapter is to describe the known exact solutions for colliding gravitational waves in which the polarization of the approaching waves is not aligned. The first solution of this type was obtained by Nutku and Halil (1977). A further generalization of this solution was attempted by Halil (1979), but this has subsequently been found to be incorrect and so will not be considered here.

13.1 The Nutku–Halil solution

The Khan–Penrose solution discussed in Chapter 3 describes the collision of impulsive waves whose polarization vectors are aligned. Nutku and Halil (1977) have generalized this solution to give one which describes the collision of impulsive gravitational waves with non-colinear polarization. This is the most simple solution of this type.

In terms of the Szekeres line element (6.20), the metric functions of this solution can be written in the form

$$\begin{aligned} e^{-U} &= 1-u^2-v^2 \\ e^{-V} &= \sqrt{\frac{(1-E)(1-\bar{E})}{(1+E)(1+\bar{E})}} \\ \sinh W &= -\frac{i(E-\bar{E})}{1-E\bar{E}} \\ e^{-M} &= \frac{1-E\bar{E}}{\sqrt{1-u^2}\sqrt{1-v^2}\sqrt{1-u^2-v^2}} \end{aligned} \tag{13.1}$$

where

$$E = e^{i\alpha}u\sqrt{1-v^2} + e^{i\beta}v\sqrt{1-u^2} \tag{13.2}$$

and α and β are constants such that $(\alpha-\beta)$ is the angle between the polarization vectors of the approaching waves. This solution can be seen to reduce to the Khan–Penrose solution (3.9) when $\alpha=\beta=0$. It may also be observed that the functions f and g are given by $f=\frac{1}{2}-u^2$, and $g=\frac{1}{2}-v^2$.

This solution has been analysed in great detail by Chandrasekhar and Ferrari (1984). Their approach commences by rewriting the field equations in the form of Ernst's equation as described in Chapter 11. It can then be seen from (11.20) that E is the associated Ernst function. Chandrasekhar and Ferrari were then able to show that the Nutku–Halil solution is obtained by the simple choice of Ernst function given by

$$E = pt + iqz \tag{13.3}$$

where p and q are real constants which satisfy the condition $p^2 + q^2 = 1$. It is always possible to use a rotation of the type (12.9) to put $\beta = -\alpha$, and in this case $p = \cos\alpha$ and $q = \sin\alpha$.

It can easily be shown that the Ernst function Z associated with this solution is given by

$$Z = \frac{1 + t\cos\alpha + iz\sin\alpha}{1 - t\cos\alpha - iz\sin\alpha}. \tag{13.4}$$

It may be observed that (13.3) is in fact the Ernst potential which leads to the Kerr solution for stationary axisymmetric space-times. This reduces to the potential for the Schwarzschild solution when $q = 0$ or $\alpha = 0$. Thus the difference in the polarization of the approaching waves in this situation can be seen to be equivalent to the rotational parameter in the Kerr solution.

Taking the Ernst functions as (13.3) or (13.4), it can be shown that the boundary conditions described in Section 7.2 can only be satisfied if $f = \frac{1}{2} - (c_1u)^2 + \ldots$, and $g = \frac{1}{2} - (c_2v)^2 + \ldots$. It is then convenient to transform the null coordinates such that $f = \frac{1}{2} - u^2$, and $g = \frac{1}{2} - v^2$. Thus the form (13.2) is obtained uniquely from the Ernst function (13.3). It also follows from this that the approaching waves necessarily contain impulsive components.

Expressions for the components of the Weyl tensor Ψ_0, Ψ_2 and Ψ_4 describing the wave and interaction components have been given by Chandrasekhar and Ferrari (1984), though in a notation which differs slightly from that adopted here. However, it can still be clearly seen that this solution is a generalization of the Khan–Penrose solution in which the approaching impulsive waves have different polarization. Apart from the initial impulsive waves on the boundaries $u = 0$ and $v = 0$, the interiors of regions II and III are flat.

It can also be seen that the singularity structure of this solution is the same as that of the Khan–Penrose solution. There is a curvature singularity on the space-like surface $u^2 + v^2 = 1$ in region IV, and there are fold singularities in regions II and III which are identical to those described in Section 8.3.

It may also be pointed out that Ernst (1986) has shown that that the Nutku–Halil solution can also be obtained using a double-Harrison (Bäcklund) transformation, with the seed metric being the isotropic Kasner metric.

13.2 The Panov solution

Panov (1979*b*) has applied the generation method of Geroch (1971), described in Section 12.3, to obtain a non-colinear solution from a colinear one.

In terms of the metric functions in (6.20), Panov started with a colinear solution U_o, V_o, and M_o of (6.22) with $W_\mathrm{o} = 0$. He then worked with the Killing vector $\xi^\mu = \delta^\mu_2$, which is (12.25) with $a = 1$ and $b = 0$. This enabled him to put $P_2 = 0$, and hence $\omega = 0$, and also from (12.30), $Q_2 = F^2 - 1$ and $Q_3 = 0$, where $F^2 = e^{V_\mathrm{o} - U_\mathrm{o}}$.

With these expressions, the metric components may be derived from (12.23), giving

$$\begin{aligned} U &= U_\mathrm{o} \\ e^{-2V} &= e^{-2V_\mathrm{o}} \Big(\cos^2\alpha + \sin^2\alpha\, e^{2(V_\mathrm{o}-U_\mathrm{o})}\Big)^2 + \sin^2 2\alpha\, P_3^2 \\ \sinh W &= \sin 2\alpha\, P_3 e^{V_\mathrm{o}} \Big(\cos^2\alpha + \sin^2\alpha\, e^{2(V_\mathrm{o}-U_\mathrm{o})}\Big)^{-1} \\ e^{-M} &= e^{-M_\mathrm{o}} \Big(\cos^2\alpha + \sin^2\alpha\, e^{2(V_\mathrm{o}-U_\mathrm{o})}\Big) \end{aligned} \tag{13.5}$$

where, from (12.28), P_3 must satisfy

$$P_{3,u} = e^{-U_\mathrm{o}}(V_{\mathrm{o},u} - U_{\mathrm{o},u})\ , \qquad P_{3,v} = -e^{-U_\mathrm{o}}(V_{\mathrm{o},v} - U_{\mathrm{o},v}) \tag{13.6}$$

or, more conveniently,

$$P_{3,f} = (f+g)V_{\mathrm{o},f} + 1, \qquad P_{3,g} = -(f+g)V_{\mathrm{o},g} - 1. \tag{13.7}$$

It is worth pointing out that, if the initial Killing vector had been taken to be $\xi^\mu = \delta^\mu_3$, which is (12.25) with $a = 0$ and $b = 1$, an equivalent transformation would have been obtained with $P_3 = 0$ and $P_2 \neq 0$. This would have given an identical set of equations, but with V being replaced by $-V$, and P_3 by P_2.

In terms of the Ernst formulation of the problem described in Chapter 11, the above transformation implies that, if Z_o is a real solution of (11.18), then a new complex solution is given by

$$Z = \cos^2\alpha\, Z_o + (1-t^2)(1-z^2)\sin^2\alpha\, Z_o^{-1} + i\sin 2\alpha\, P_3 \tag{13.8}$$

where $P_3(t,z)$ must satisfy the equations

$$\begin{aligned} P_{3,t} &= -(1-z^2)(\log Z_o)_{,z} - z \\ P_{3,z} &= -(1-t^2)(\log Z_o)_{,t} - t. \end{aligned} \tag{13.9}$$

Panov then took the initial solution to be the totally general solution of Szekeres (1972), which can be considered to be equivalent to the general solution given here by (10.16). He has thus determined a general class of non-colinear solutions, and he was able to show that, when the appropriate boundary conditions have been applied, the approaching waves necessarily have variable polarization.

It has already been shown that the new solution obtained using a Geroch transformation will automatically satisfy the required boundary conditions if these are satisfied by the initial solution. In this case, it can also be seen that the new and initial solutions have the same singularity structure.

13.3 The Chandrasekhar–Xanthopoulos solution

Chandrasekhar and Ferrari (1984) have shown that the associated Ernst function which is contained in the Nutku–Halil solution can be written as $E_o = pt + iqz$ (13.3), where $p^2 + q^2 = 1$. Now, it is well known (see Chandrasekhar, 1983) that this is the Ernst potential that leads to the Kerr solution for stationary axisymmetric space-times. However, as described in Section 12.4, Chandrasekhar and Ferrari have also shown that, for the colliding plane wave situation, the Ernst function may be regarded either (a) as containing the metric functions, or (b) as a potential for those functions.

When the Ernst function (13.3) is considered as containing the metric functions, it leads to the Nutku–Halil solution described above. However, when the same function is considered as a potential, it then leads to the Kerr solution. Thus part of the the Kerr space-time must also be considered as a solution of the colliding plane wave equations, though in this situation the coordinates have a different interpretation.

This alternative interpretation of part of the Kerr space-time has been described in detail by Chandrasekhar and Xanthopoulos (1986*b*).

Essentially it may be considered to have been derived from the Nutku–Halil solution using the Neugebauer–Kramer involution described in Section 12.4.

We start here with the Ernst potential given by $E = pt + iqz$, so that

$$Z = \Psi + i\Phi = \frac{1 + pt + iqz}{1 - pt - iqz} \tag{13.10}$$

where $p^2 + q^2 = 1$. According to (12.36), the first metric function is then given by

$$\begin{aligned} \chi &= \sqrt{1 - t^2}\sqrt{1 - z^2}\,\Psi^{-1} \\ &= \sqrt{1 - t^2}\sqrt{1 - z^2}\left(\frac{(1 - pt)^2 + q^2 z^2}{1 - p^2 t^2 - q^2 z^2}\right) \end{aligned} \tag{13.11}$$

and, from (12.34), ω is given by

$$\omega_z = \frac{(1 - t^2)}{\Psi^2}\,\Phi_t, \qquad \omega_t = \frac{(1 - z^2)}{\Psi^2}\,\Phi_z \tag{13.12}$$

where

$$\Phi = \frac{2qz}{(1 - pt)^2 + q^2 z^2}. \tag{13.13}$$

With this, (13.12) becomes

$$\omega_z = \frac{4pq(1 - t^2)(1 - pt)z}{(1 - p^2 t^2 - q^2 z^2)^2}, \qquad \omega_t = \frac{2q(1 - z^2)\big((1 - pt)^2 - q^2 z^2\big)}{(1 - p^2 t^2 - q^2 z^2)^2} \tag{13.14}$$

which can immediately be integrated to give

$$\omega = \frac{2}{q}\left(\frac{p(1 - t^2)(1 - pt)}{(1 - p^2 t^2 - q^2 z^2)} + t + c\right) \tag{13.15}$$

where c is an arbitrary constant.

Chandrasekhar and Xanthopoulos have chosen the constant of integration in (13.15) such that $\omega = 0$ when $t = 1$. The reason for this choice is associated with the convenience of aligning the coordinate directions with the shear axes on the surface $f + g = 0$ on which the two waves mutually focus each other. Accordingly we set

$$c = -1. \tag{13.16}$$

A variation of this constant is equivalent to making an Ehlers transformation as described in (12.16–18). With (13.16), ω may now be written in either of the forms

$$\begin{aligned}\omega &= -\frac{2q}{(1+p)}\frac{(1-t)(1-pt-z^2-pz^2)}{(1-p^2t^2-q^2z^2)}\\ &= \frac{2q}{p(1+p)} - \frac{2q(1-z^2)(1-pt)}{p(1-p^2t^2-q^2z^2)}.\end{aligned} \tag{13.17}$$

For purposes of simplification, it is convenient to introduce the terms

$$\begin{aligned}X &= (1-pt)^2+q^2z^2\\ Y &= 1-p^2t^2-q^2z^2 \;=\; p^2(1-t^2)+q^2(1-z^2).\end{aligned} \tag{13.18}$$

In view of (12.43) and (13.1), in this case we must have

$$e^{-U} = \sqrt{1-t^2}\sqrt{1-z^2} \;=\; 1-u^2-v^2. \tag{13.19}$$

The metric function M can also be obtained from (12.42) and (13.1), giving

$$e^{-M} = \frac{X}{\sqrt{1-u^2}\sqrt{1-v^2}}. \tag{13.20}$$

These expressions now complete all the metric functions contained in the line element (11.4), which may now be written as

$$\begin{aligned}\mathrm{d}s^2 = {}& \frac{2X\mathrm{d}u\,\mathrm{d}v}{\sqrt{1-u^2}\sqrt{1-v^2}} - \frac{Y}{X}\left(\mathrm{d}x - \frac{2q}{p(1+p)}\mathrm{d}y\right)^2\\ &- \frac{4q(1-z^2)(1-pt)}{pX}\left(\mathrm{d}x - \frac{2q}{p(1+p)}\mathrm{d}y\right)\mathrm{d}y\\ &- \frac{(1-z^2)}{p^2XY}\Big(p^2(1-t^2)X^2+4q^2(1-z^2)(1-pt)^2\Big)\mathrm{d}y^2.\end{aligned} \tag{13.21}$$

After some rearrangement, this may be written in the more convenient form

$$\begin{aligned}\mathrm{d}s^2 = {}& \frac{X}{2}\left(\frac{\mathrm{d}t^2}{1-t^2} - \frac{\mathrm{d}z^2}{1-z^2}\right) - \frac{Y}{X}\left(\mathrm{d}x - \frac{2q}{p(1+p)}\mathrm{d}y\right)^2\\ &- \frac{4q(1-z^2)(1-pt)}{pX}\left(\mathrm{d}x - \frac{2q}{p(1+p)}\mathrm{d}y\right)\mathrm{d}y\\ &- \frac{(1-z^2)}{p^2X}\Big(\big((1-pt)^2+q^2\big)^2+p^2q^2(1-t^2)(1-z^2)\Big)\mathrm{d}y^2\end{aligned} \tag{13.22}$$

in which Y does not appear in a denominator.

In order to show that this is indeed the Kerr solution, we first relabel the parameters by putting

$$p = -\sqrt{M^2 - a^2}/M, \qquad q = a/M \tag{13.23}$$

which satisfies the condition $p^2 + q^2 = 1$ for arbitrary constants M and a. It is also appropriate to introduce the new coordinates r and θ instead of t and z given by

$$t = (r - M)/\sqrt{M^2 - a^2}, \qquad z = \cos\theta \tag{13.24}$$

where it may be noticed that, using the previous notation, $\lambda = \pi/2 - \theta$. The remaining coordinates may also be transformed by putting

$$\tau = -\sqrt{2}\,M\left(x - \frac{2q}{p(1+p)}\,y\right), \qquad \phi = \frac{\sqrt{2}\,M}{\sqrt{M^2 - a^2}}\,y. \tag{13.25}$$

It is also convenient to introduce the definitions

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2Mr + a^2. \tag{13.26}$$

With these substitutions the line element (13.22) may then be written in the form

$$\begin{aligned} 2M^2\mathrm{d}s^2 = {} & \left(1 - \frac{2Mr}{\rho^2}\right)\mathrm{d}\tau^2 - \frac{4aMr}{\rho^2}\sin^2\theta\,\mathrm{d}\tau\mathrm{d}\phi \\ & - \left(r^2 + a^2 - \frac{2a^2Mr}{\rho^2}\right)\sin^2\theta\,\mathrm{d}\phi^2 - \rho^2\left(\frac{1}{\Delta}\mathrm{d}r^2 + \mathrm{d}\theta^2\right) \end{aligned} \tag{13.27}$$

which after rescaling $\mathrm{d}s$ is the standard form of the Kerr solution.

For this solution to describe the interaction region of colliding waves, the coordinates must satisfy the inequality $|z| < t \le 1$. With (13.24), this implies that

$$-(M^2 - a^2)\sin^2\theta < \Delta \le 0 \tag{13.28}$$

which is satisfied only by the region of the Kerr space-time that is inside the ergo-sphere. This is not unexpected, as this is the only region of Kerr space-time in which the Killing vectors are both space-like.

The plane on which the two waves collide is given by $t = 0$, $z = 0$, or by $r = M$, $\theta = \pi/2$ which is between the two horizons. The hypersurface on which they mutually focus each other, which is given by $f + g = 0$ or $t = 1$, is here given by $\Delta = 0$ which determines the horizons of the

Kerr solution. Thus, the curvature singularity which usually occurs when $f+g=0$, in this case, is replaced by a coordinate singularity which is normally interpreted as a horizon of the Kerr solution.

The components of the Weyl tensor inside region IV can be derived using (11.10) and have been evaluated by Chandrasekhar and Xanthopoulos. The space-time is of algebraic type D and, in particular, it may be noted that

$$\Psi_2 = \frac{1}{(1-pt-iqz)^3} = \frac{M^3}{(r-ia\cos\theta)^3}. \tag{13.29}$$

The curvature is clearly non-singular as $t\to 1$ except in the aligned case $p=1$, $q=0$ (or $a=0$). In this case only, the surface $t=1$ corresponds to a curvature singularity.

In the interaction region t is a future pointing time-like coordinate. It thus follows from (13.24) that r is also a future pointing time-like coordinate which increases from the collision at $r=M$ to the outer horizon at $r=M+\sqrt{M^2-a^2}$. In this case a possible further extension beyond the horizon would present no problems.

It may be noted, however, that if the signs of the expressions for t, p and q in (13.24) and (13.23) are changed, the Kerr metric (13.27) is still obtained but the orientation of r is altered. In this case, the collision at $r=M$ is followed by the inner horizon at $r=M-\sqrt{M^2-a^2}$ and any extension would lead to a future time-like curvature singularity at $r=0$.

The difference between these two solution has been further analysed by Hoenselaers and Ernst (1990). It is the latter case that has been described by Chandrasekhar and Xanthopoulos (1986*b*) in which there is an extension beyond the horizon at $t=1$ which contains a time-like singularity analogous to the ring singularity[1] of the Kerr solution at $r=0$, $\theta=\pi/2$. Since this singularity is time-like, it would be missed by most observers. In the other case, which has been given explicitly above, there is an analytic extension which is the asymptotically flat exterior Kerr solution. It may be pointed out, however, that in both cases the extension beyond the horizon is not unique.

When considering the question of the character of the coordinate singularity at $t=1$, Chandrasekhar and Xanthopoulos (1986*b*) have given a transformation by which it can be removed. They have found it convenient to introduce a decreasing time-like coordinate given by $s=1-u^2-v^2$. The transformation

$$\xi = s\,e^{+x/q}, \qquad \zeta = s\,e^{-x/q} \tag{13.30}$$

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[1] For a description of the structure of the Kerr solution, see Hawking and Ellis (1973), or Chandrasekhar (1983).

then removes the coordinate singularity at $s = 0$, or $t = 1$. This provides an analytic extension beyond the Cauchy horizon which is another part of Kerr space-time. This extension, in one case, is followed by a time-like curvature singularity which corresponds to the source of the Kerr solution.

It may be seen that the singularity structure of this solution is very similar to that of the degenerate Ferrari–Ibañez solution described in detail in Section 10.5. In fact this solution reduces to either of the two cases of the degenerate Ferrari–Ibañez solution in the limit as $q \to 0$, which is the limit as the polarization of the approaching waves become aligned. The non-singular case occurs when $p = -1$. It may be noted that in this limit $a \to 0$, the rotation of the Kerr solution vanishes, and the solution reduces to the same part of the Schwarzschild space-time as the degenerate Ferrari–Ibañez solution.

It is now appropriate to consider the extension of the solution into the prior regions I, II and III which describe the approaching waves that give rise to this particular interaction. The metric (13.21) may easily be extended into region II simply by replacing both t and z by u. It turns out to be convenient to make the coordinate transformation

$$\tilde{x} = x + \frac{2q}{1+p} y. \tag{13.31}$$

With this, the line-element in region II takes the form

$$\mathrm{d}s^2 = \frac{2X}{\sqrt{1-u^2}} \mathrm{d}u\,\mathrm{d}v - (1-u^2)\left(X\mathrm{d}y^2 + \frac{1}{X}(\mathrm{d}\tilde{x} - 2qu\,\mathrm{d}y)^2 \right) \tag{13.32}$$

where $X = 1 - 2pu + u^2$. The line-element for region III is identical to (13.32) except that v replaces u. Both these line-elements are then continuous with the Minkowski space (3.6) in region I.

The plane wave metric given by (13.32) has the single component of the curvature tensor given by

$$\Psi_4 = -(p - iq)\delta(u) - \frac{3\,(X - 2iqu)}{X^4\sqrt{X^2 + 4q^2u^2}} \frac{(1 - pu - iqu)^3}{(p + iq)^2} \Theta(u). \tag{13.33}$$

It may immediately be observed that the approaching waves have variable polarization except in the aligned limit $q \to 0$. They include an impulsive component and a step component.

As in the degenerate Ferrari–Ibañez solution, the singularity structure is thus complicated by the fact that the solution contains impulsive waves on the wave fronts $u = 0$ and $v = 0$ which form the boundaries of the interaction region. The presence of these wave components introduces additional distribution valued singularities at the points $u = 0$, $v = 1$ and $v = 0$, $u = 1$. As described in Chapter 8, these point singularities ensure the existence of fold singularities on the surfaces $u = 1$ and $v = 1$ in regions II and III respectively.

The profiles for the approaching waves in both cases have been described in more detail by Hoenselaers and Ernst (1990). They have shown that, in the case where the Cauchy horizon corresponds to the inner Kerr horizon, the amplitude diverges towards the fold singularity which is thus a non-scalar curvature singularity. On the other hand, when the Cauchy horizon corresponds to the outer Kerr horizon, the amplitude decays towards a quasiregular fold singularity.

13.4 Other solutions

It may be noticed that the solutions presented in the previous two sections have been obtained using the generating techniques described in Sections 12.3 and 12.4 respectively. Other techniques may also be used, both in isolation and in various combinations. Since they involve the same Ernst equation, it is clear that all the solution generating techniques that have been developed for stationary axisymmetric space-times in recent years can also be applied to colliding plane wave situations.

A number of papers have recently been published which present new colliding wave solutions using some of these techniques, and it is likely that many more such papers will appear. Most of these concentrate on a description of a particular technique and how it may be adapted to the colliding wave situation. At present, however, there is less emphasis on the physical significance of the particular solution generated. Nevertheless, solutions with a number of parameters have been obtained, and general classes of solutions have been defined.

Ideally, one would like to be able to specify the particular solution corresponding to an arbitrary set of initial conditions. Such techniques are available in the axisymmetric case, and may soon be extended to the colliding wave situation. The initial attempts at the solution of this problem are mentioned in the next chapter.

So far most of the explicit solutions that have been presented have involved impulsive wave components. In the notation of (7.11), they have used $n_1 = n_2 = 2$. Much less attention has been given to smooth-fronted waves of finite duration.

Some further general aspects of cases involving impulsive components have been clarified by Ferrari (1988). In particular, it has been shown that the proper time between the collision and the subsequent singularity is inversely proportional to the square root of the amplitudes of the impulsive components of the approaching waves. This time is also affected by their relative polarization and is a minimum when the approaching waves are colinear. This result also applies to solutions which contain a Cauchy horizon rather than a curvature singularity. In addition, Ferrari has also shown that the shock waves which accompany the impulsive waves only affect the rate at which the Weyl scalars diverge on the singularity.

Having made these general remarks, we may now briefly review the new exact solutions that have been presented. The generation techniques themselves will not be described in detail here.

First, there is the important class of solutions that has been obtained by Ferrari, Ibañez and Bruni (1987*a,b*) using the inverse scattering method developed by Belinskii and Zakharov (1978, 1979) and Carr and Verdaguer (1983). Using this method, they have obtained a two-parameter class of colliding wave solutions with non-aligned polarization. This in fact is a generalization of the soliton solution of Ferrari and Ibañez (1987*b*) described in Sections 10.4 and 10.5.

The Ferrari–Ibañez–Bruni solution has been obtained using as seed the Kasner metric. They utilize the Belinskii–Zakharov soliton technique with two real poles. The resulting metric for region IV is singular on the space-like surface $f+g=0$, except in a particular limiting case. By extending the solution back into regions II and III, it can be seen that the approaching waves must have both an impulsive component and a step component with variable polarization.

The limiting case of the Ferrari–Ibañez–Bruni solution is particularly interesting. It is of type D, and can be shown to be part of the Taub–NUT solution[2] in the Taub region where there are two space-like Killing vectors. The line element can be written in the form

$$\begin{aligned}\mathrm{d}s^2 &= C(1+2p\sin\psi+\sin\psi^2)(\mathrm{d}\psi^2-\mathrm{d}\lambda^2)\\ &\quad-\left(\frac{1-\sin^2\psi}{1+2p\sin\psi+\sin^2\psi}\right)(\mathrm{d}x-2q\sin\lambda\mathrm{d}y)^2\\ &\quad-\cos^2\lambda(1+2p\sin\psi+\sin^2\psi)\mathrm{d}y^2\end{aligned}\qquad(13.34)$$

where p and q are constants satisfying $p^2+q^2=1$. It can immediately be seen that, when $q=0$ and $p=\pm1$, this further reduces to the two colinear degenerate Ferrari–Ibañez solutions that have already been considered in Section 10.5. The structure of this solution has been further examined by Ferrari and Ibañez (1988).

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[2] Newman, Tamburino and Unti, (1963); see also Hawking and Ellis (1973) section 5.8.

Ernst, García-Díaz and Hauser (1987*a*) have shown that an Ehlers transformation (12.13) applied to the Ernst function of the Nutku–Halil solution (13.3), together with a simple coordinate transformation, yields a non-colinear generalization of the Ferrari–Ibañez soliton solution that was described in Section 10.4 but only with even integer values of the parameter a. This generalization is in fact contained in the solution of Ferrari, Ibañez and Bruni (1987*a,b*).

A further generalization extending the Ferrari–Ibañez–Bruni solution was reported by Ernst, García-Díaz and Hauser (1987*a,b*), with a full derivation given in Ernst, García-Díaz and Hauser (1988). This solution contains three arbitrary parameters. In stating this solution, it is convenient first to define the function

$$\begin{aligned} T(a,\alpha,\beta) = {}& \frac{\sqrt{1-t^2}}{2}\left[(p+p')\left(\frac{1-t}{1+t}\right)^{a/2} + (p-p')\left(\frac{1+t}{1-t}\right)^{a/2}\right] \\ & + \frac{i\sqrt{1-z^2}}{2}\left[(q+q')\left(\frac{1-z}{1+z}\right)^{a/2} + (q-q')\left(\frac{1+z}{1-z}\right)^{a/2}\right] \end{aligned} \tag{13.35}$$

where a, α and β are the arbitrary parameters, and where

$$p = \cos\alpha, \qquad q = \sin\alpha, \qquad p' = \cos\beta, \qquad q' = \sin\beta. \tag{13.36}$$

With this definition, the Ernst function for this new solution may be written as

$$Z(a,\alpha,\beta) = (1-t^2)^{a/2}(1-z^2)^{a/2}\frac{T(a+1,\beta,\alpha)}{T(a-1,\beta,\alpha)}. \tag{13.37}$$

When $a = 0$, this reduces to the Nutku–Halil solution. With $|a| = 1$, one obtains the Kerr–NUT solution with α being the NUT parameter. The subcase with $|a| = 1$ and $\beta = 0$ becomes the Schwarzschild–NUT space-time.

Transforming (13.37) using the Neugebauer–Kramer involution produces another three-parameter family of solutions with the alternative Ernst function given by

$$Z(a,\alpha,\beta) = (1-t^2)^{(1-a)/2}(1-z^2)^{(1-a)/2}\frac{T(a-2,\alpha,\beta)}{T(a,\alpha,\beta)}. \tag{13.38}$$

With $a = 0$, this reduces to the Chandrasekhar–Xanthopoulos solution.

Ernst, García-Díaz and Hauser (1987*b*) have also found a simpler realization $G_0(\Sigma_1)$ of the Geroch group of transformations relating colliding wave solutions. With this , they have obtained a generalization of the above solution in the case when $a = 2$. A further generalization of the $a = 3$ solution is also obtained using a Neugebauer–Kramer involution.

It was pointed out by Ernst (1986) that the Nutku–Halil solution can be derived from the isotropic ($n = 0$) Kasner metric by applying a double-Harrison (Bäcklund) transformation. The Kerr space-time of the Chandrasekhar–Xanthopoulos solution can also be derived from Minkowski space using a double-Harrison transformation. It therefore seemed reasonable that other colliding wave solutions may be generated by applying double-Harrison transformations to other Kasner metrics. This procedure has indeed been successful. It was used by Ernst, García-Díaz and Hauser (1988) to determine the solution described above in (13.35) to (13.38).

The effect of a double-Harrison transformation can be deduced by any number of competing techniques. Naturally, however, Ernst, García-Díaz and Hauser preferred to use the so-called Hauser–Ernst homogeneous Hilbert problem approach (see also Ernst 1988).

Finally, it may be mentioned that Hassan, Feinstein and Manko (1990) have recently obtained another particular solution describing the non-colinear case using the generating technique of Gutsunaev and Manko (1988) as described here in (12.56) and (15.57). Further solutions could easily be obtained using this method.